Tutorial on Distributed Optimization

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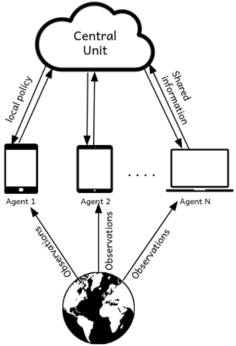
Motivation



Centralized

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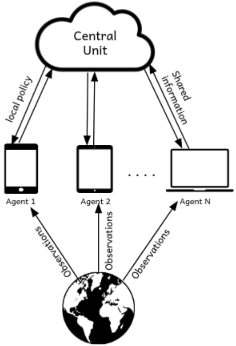

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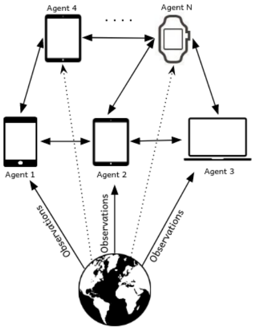
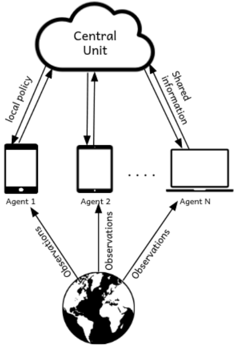


Networked Agents

Motivation



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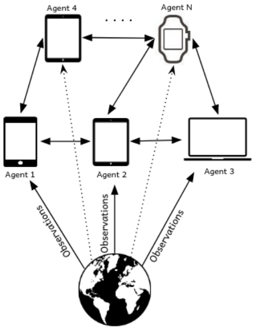

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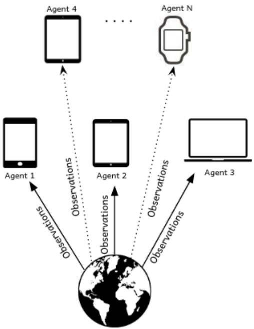

Fully Decentralized

Motivation

- Fully decentralized learning scheme

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- Fully decentralized learning scheme
  - Non-stationarity problem
  - Local solutions may not necessarily lead to global ones
  - Non-cooperative setting

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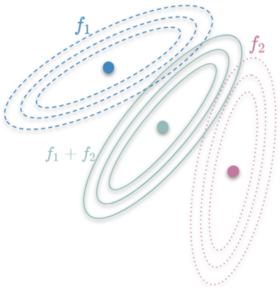
  - Existence of a central coordinator
- Networked Optimization scheme
  - Existence of a communication structure to enable coordination
  - Information exchange with neighboring nodes

Motivation

$$f(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^{N} f^i(\mathbf{x}) \tag{1}$$



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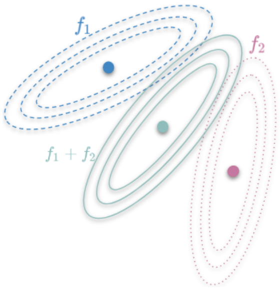

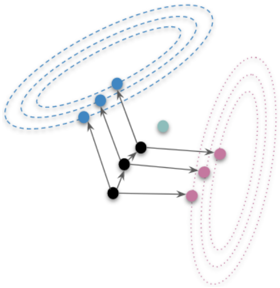

$x_k = x_{k-1} - \alpha_k \sum_{i=0}^{N} g^i(x_{k-1})$

Motivation

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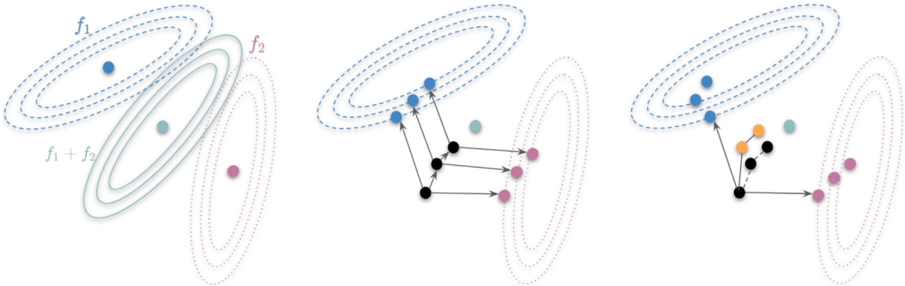
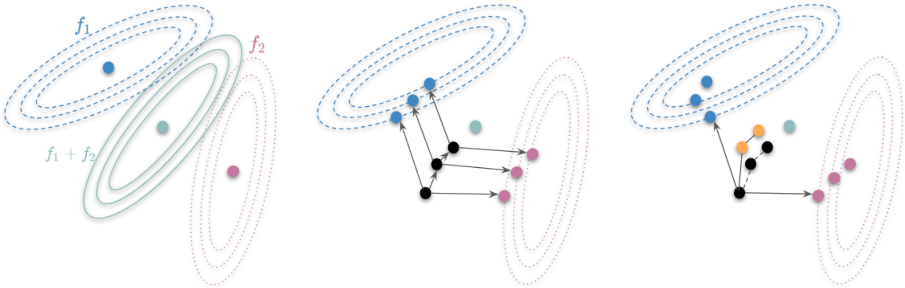


$x_k = x_{k-1} - \alpha_k \sum_{i=0}^{N} g^i(x_{k-1})$

$x_k^i = x_{k-1}^i - \alpha_k^i g^i(x_{k-1}^i)$

Motivation

$$\mathbf{f}(\mathbf{x}) = \frac{1}{N}\sum_{i=1}^{N} f^i(x) \tag{2}$$



slow (only 1 update)

Potentially faster but may not converge

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Decomposible Problems

- Optimization problems with *coupling* variables

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Decomposible Problems

- Optimization problems with *coupling* variables

$$\max_{\mathbf{x}} f(\mathbf{x}) = \frac{1}{N}\sum_{i=1}^{N} f^i(\mathbf{x})$$

- Optimization problems with *coupling* constraints

$$\max_{x_i} \frac{1}{N}\sum_{i=1}^{N} f^i(x_i) \text{ s.t } C(\mathbf{x}) = B$$

Decomposition: Motivation

Power allocation in wireless communication

$$\max_{p_i} \sum_{i=0}^{N} \log(1 + \frac{g_i p_i}{\sigma^2})$$

$$\text{s.t} \sum_{i=0}^{N} p_i = P_{\max} \longleftarrow \text{coupling constraint}$$

Decomposition: Motivation

Decomposition benefits for MAS:

- Computational Scalability and flexibility
- Tractability
- Privacy
- ....

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Dual Ascent Algorithm

Consider the problem

$$\min_{\mathbf{x}} f(\mathbf{x}) \text{ s.t } A\mathbf{x} = \mathbf{b}$$

- **Lagrangian**

$$L(\mathbf{x}, \boldsymbol{u}) = f(\mathbf{x}) + (A\mathbf{x} - \mathbf{b})^T \boldsymbol{u}$$

- **Dual function**

$$g(\boldsymbol{u}) = \min_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{u})$$

- **Conjugate function/Fenchel conjugate**

$$f^\star(\mathbf{y}) = \max_{\mathbf{x}}[\mathbf{y}^T\mathbf{x} - f(\mathbf{x})]$$

$$-f^\star(\mathbf{y}) = \min_{\mathbf{x}}[f(\mathbf{x}) - \mathbf{y}^T\mathbf{x}]$$

$$g(\boldsymbol{u}) = -f^\star(-A^T\boldsymbol{u}) - \mathbf{b}^T\boldsymbol{u}$$

Reminder: Conjugate functions

- If f is closed and convex, $(f^\star)^\star = f$ and

$$x \in \partial f^\star \iff y \in \partial f \iff x \in \arg\min_z f(z) - y^T z \tag{3}$$

- If f is strictly convex, $\nabla f^\star(y) = \arg\min_z f(z) - y^T z$
- $f^\star$ is always convex even if f is not convex (i.e affine in y)

Dual Ascent Algorithm

Dual Problem:

$$\max_{\boldsymbol{u}} g(\boldsymbol{u}) = min_u f^\star(-A^T\boldsymbol{u}) + \mathbf{b}^T\boldsymbol{u}$$

$$\implies \partial g(\boldsymbol{u}) = A\partial f^\star(-A^T\boldsymbol{u}) - \mathbf{b}$$

$$\implies \partial g(\boldsymbol{u}) = A\mathbf{x}^* - \mathbf{b} \text{ where } \mathbf{x}^* \in \arg\min_{\mathbf{x}} f(\mathbf{x}) + \boldsymbol{u}^T A\mathbf{x} \quad (3)$$

Apply sub/gradient methods to the dual : $u_{k+1} = u_k + t_k\partial g(u_k)$

Dual Ascent Algorithm

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Apply sub/gradient methods to the dual : $u_{k+1} = u_k + t_k \partial g(u_k)$

Algorithm 2: Dual Sub-gradient Method

Start with an initial dual λ_0 and repeat for K iterations; t_0 initial learning rate

for $k = 1, \ldots, K$ **do**

$\quad \mathbf{x}_k \in \arg\min_{\mathbf{x}} f(\mathbf{x}) + \boldsymbol{u}_{\boldsymbol{k-1}}^T A\mathbf{x}$

$\quad \boldsymbol{u}_{\boldsymbol{k}} = \boldsymbol{u}_{\boldsymbol{k-1}} + t_k(A\mathbf{x}_k - \mathbf{b})$

end

Separable Problem Example

Primal Problem

$$\min_{x_1,x_2} f_1(x_1) + f_2(x_2)$$
$$\text{s.t } x_1 + x_2 \leq P$$

Dual Problem

$$\max_u -f_1^\star(-u) - f_2^\star(-u) - Pu$$
$$\text{s.t } u \geq 0$$

The Lagrangian is separable in f:

$$L(x_1, x_2, u) = f_1(x_1) + ux_1 + f_2(x_2) + ux_2 - Pu$$
$$g(u) = g_1(u) + g_2(u) - Pu; \quad g_i(u) = \min_{x_i} f_i(x_i) + ux_i = -f_i^\star(-u), i = 1, 2$$

Applying Dual Ascent will result in

$$u_{k+1} = u_k + t_k \text{Proj}_{u\geq 0}\left\{x_k^1 + x_k^2 - b\right\} = u_k + t_k \max\left\{x_k^1 + x_k^2 - b, 0\right\}$$
$$x_k^i = \arg\min_{x^i} f_i(x^i) + u_{k-1}x^i, \quad i = 1, 2 \implies \text{ decoupled}$$

Dual Decomposition

More generally,

$$\min_{\mathbf{x}_i} \sum_{i=1}^{B} f_i(\mathbf{x}_i) \qquad \text{separable}$$
$$\text{s.t } A\mathbf{x} = \mathbf{b} \qquad \text{not separable}$$

Dual Decomposition

More generally,

$$\min_{\mathbf{x}_i} \sum_{i=1}^{B} f_i(\mathbf{x}_i) \quad \text{separable}$$
$$\text{s.t } A\mathbf{x} = \mathbf{b} \quad \text{not separable}$$

Algorithm 4: Dual Decomposition Method (coupled constraints)

Start with an initial dual λ_0 and repeat for K iterations; t_0 initial learning rate **for** $k = 1, \ldots, K$ **do**

$\quad \mathbf{x}_k^i = \arg\min_{\mathbf{x}^i} f_i(\mathbf{x}^i) + \boldsymbol{u}_{\boldsymbol{k-1}}^T A^i \mathbf{x}^i$

$\quad \boldsymbol{u}_{\boldsymbol{k}} = \boldsymbol{u}_{\boldsymbol{k-1}} + t_k(\sum_{i=1}^{B} A^i \mathbf{x}_k^i - \mathbf{b})$

end

- The minimization problems w.r.t to x_i are independent
- Coordination is done through adjusting u to maximize $g(u)$

Dual Decomposition for coupling variables

Primal Problem

$$\min_{x_1,x_2,y} f_1(x_1,y) + f_2(x_2,y)$$

Transform the problem to

$$\begin{array}{ll} \min_{x_1,x_2,y_1,y_2} f_1(x_1,y_1) + f_2(x_2,y_2) & : y_1, y_2 \text{ are } \textbf{local copies} \text{ of } y \\ \text{s.t } y_1 = y_2 & : \textbf{consensus constraint} \end{array}$$

Algorithm 5: Dual Decomposition for coupling variables

Start with an initial dual λ_0 and repeat for K iterations; t_0 initial learning rate

for $k = 1 \ldots K$ **do**

$\quad x_k^i, y_k^i = \arg\min_{x^i,y^i} f_i(x^i,y^i) + u_{k-1}y^i, \quad i = 1,2$

$\quad u_k = u_{k-1} + t_k(y_k^1 - y_k^2)$

end

But, the convergence to a feasible iterates (i.e $y_2 - y_1 \neq 0$) is not always guaranteed.

Discussion

- The dual variable u is equivalent to a price of a shared resource
- If the resource is over-utilized (i.e $P - x_1 - x_2 < 0$), the price is increased and decreased otherwise.
- But, the price never gets negative !
- A centralized unit is needed for coordination to update u

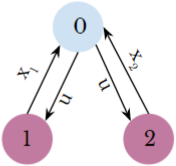

Figure: Price Coordination interpretation of DD (see slide 16)

Discussion

- **Subgradient** methods are slow, step size selection is difficult
- **Gradient** methods are applicable if the dual function is differentiable (which is not always the case, requires strict convexity of the primal function)
- The convergence guarantees of the Dual Ascent method do not necessarily guarantee the convergence of the primal (i.e Slater's Theorem)

Discussion

Problem 1

$$\min_{x_1, x_2} 2|x_1 - 2| + 4|x_2 - 4|$$
$$\text{s.t } x_1 + x_2 = 5$$
$$0 \leq x_1 \leq 10, 0 \leq x_2 \leq 10$$

The optimal solution : $x_1^* = 1, x_2^* = 4$ with optimal value is $f^* = 2$

Problem 2

$$\min_{x_1, x_2} 2(x_1 - 2)^2 + 4(x_2 - 4)^2$$
$$\text{s.t } x_1 + x_2 = 5$$
$$0 \leq x_1 \leq 10, 0 \leq x_2 \leq 10$$

The solution for this problem is $x_1^* = 4/3, x_2^* = 11/3$ and the optimal value is $f^* = 4/3$.

Discussion

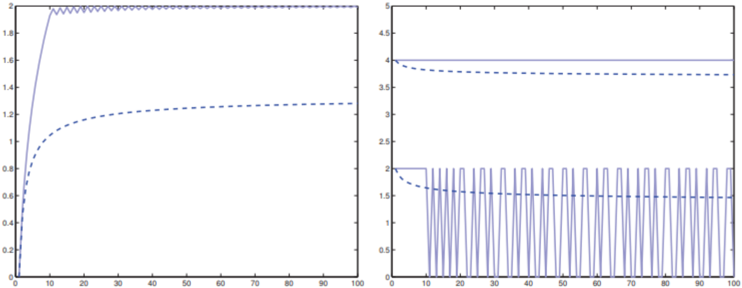

Figure: When the dual function is non-smooth, the dual objective (left, full lines) converges while the primal iterates (right, full lines) do not. When the dual function is differentiable, both objective and iterates converge asymptotically (dashed lines) [1]

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Augmented Lagrangian Methods (ALM)/ Methods of Multipliers

Adds a penalty term to the primal objective to ensure the smoothness of the dual function

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Considers the modified problem, for a parameter $\rho > 0$

$$\begin{aligned} &\min_{\mathbf{x}} f(\mathbf{x}) + \frac{\rho}{2}||A\mathbf{x} - \mathbf{b}||_2^2 \\ &\text{s.t } A\mathbf{x} = \mathbf{b} \end{aligned} \tag{4}$$

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Uses the augmented Lagrangian

$$L_\rho(\mathbf{x}, \boldsymbol{u}) = f(\mathbf{x}) + \boldsymbol{u}^T(A\mathbf{x} - \mathbf{b}) + \frac{\rho}{2}||A\mathbf{x} - \mathbf{b}||_2^2 \quad (5)$$

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Repeats, $k = 1 \dots K$

$$\begin{aligned} \mathbf{x}_k &= \arg\min_{\mathbf{x}} f(\mathbf{x}) + \boldsymbol{u}_{k-1}^T A\mathbf{x} + \frac{\rho}{2}||A\mathbf{x} - \mathbf{b}||_2^2 \\ \boldsymbol{u}_k &= \boldsymbol{u}_{\boldsymbol{k-1}} + \rho(A\mathbf{x}_k - \mathbf{b}) \end{aligned}$$

ALM convergence

- Optimality conditions

$$\text{Primal feasibility } Ax^* - b = 0; \text{ stationarity } \partial f(x^*) + A^T u^* = 0$$

- x_k minimizes $L_\rho(x, y_k)$

$$\mathbf{x}_k = \arg\min_{\mathbf{x}} f(\mathbf{x}) + \boldsymbol{\lambda}_{k-1}^T A\mathbf{x} + +\frac{\rho}{2}||A\mathbf{x} - \mathbf{b}||_2^2$$
$$\Longrightarrow 0 \in \partial f(\mathbf{x}_k) + A^T \underbrace{(\boldsymbol{\lambda}_{k-1} + \rho(A\mathbf{x}_k - \mathbf{b}))}_{\boldsymbol{\lambda}_k}$$

- the dual update makes the pair (x_k, y_k) satisfy the **stationarity** condition
- Under mild conditions, $A\mathbf{x}_k - \mathbf{b} \longrightarrow 0$ as $k \longrightarrow \infty$, KKT conditions are satisfied in the limit
- x_k and u_k converge to solutions

ALM

- The objective function stays the same
- The primal objective because strongly convex if A is full rank (f is a convex function)
- The Dual function becomes smooth $\Longrightarrow$ gradient ascent
- Better convergence guarantees compared to Dual Ascent (f can be non-differentiable, take on value $+\infty$)
- The primal objective is no longer decomposible because of the quadratic term

Proximal Operator

$$\text{prox}_{\rho f}(\tilde{x}) = \arg\min_{x} f(x) + \frac{1}{2\rho}||x - \tilde{x}||^2$$

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Proximal Point Algorithm (PPA)[2]

$$x_{t+1} = \text{prox}_{\rho f}(x_t) = (I + \rho \partial f)^{-1}(x_t)$$

Augmented Lagrangian and Proximal Point Methods

$$\textbf{Primal}: \min_x f(x) + g(Ax)$$
$$\textbf{Dual}: \max_u -f^\star(-A^T u) - g^\star(u) = \max_u -H(u)$$

Applying the PPA in the dual problem:

$$\begin{aligned} u_{t+1} &= \text{prox}_{\rho H}(u_t) \\ &= \arg\min_u f^\star(-A^T u) + g^\star(u) + \frac{1}{2\rho}||u - u_t||_2^2 \end{aligned}$$

Is equivalent to

$$u_{t+1} = u_t + \rho(A\tilde{x} - \tilde{y}) \text{ where}$$
$$(\tilde{x}, \tilde{y}) = \arg\min_{x,y}(f(x) + g(y) + u_t^T(Ax - y) + \frac{\rho}{2}||Ax - y||_2^2)$$

Hence, the augmented Lagrangian in the primal is the same as PPA in the dual

Example

$$\min_{\mathbf{x}} \sum_{i=1}^{N} f_i(\mathbf{x})$$

$$\begin{cases} \min_{\boldsymbol{x}_i} \sum_{i=1}^{N} f_i(\boldsymbol{x}_i) \\ \text{s.t } \mathbf{x}_i = \mathbf{x} \quad i = 1 \dots N \end{cases}$$

$g = \mathbb{1}_{\boldsymbol{x}}$ (indicator function), $A = I_N$

The augmented Lagrangian method

$$(x^i_{k+1}, x_{k+1}) = \arg\min_{x_i, x} (f_i(x_i) + u_t^T (x_i - x) + \frac{\rho}{2} ||x_i - x||_2^2)$$

$$u^i_{k+1} = u^i_k + \rho(x^i_{k+1} - x_{k+1})$$

Coupling between $\boldsymbol{x}$ and $\boldsymbol{x}_i$

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Combines the best of both methods (decomposibility and convergence guarantees) [3]

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Consider a problem of the form:

$$\min_{\mathbf{x},\mathbf{y}} f(\mathbf{x}) + g(\mathbf{y}) \qquad (6)$$
$$\text{s.t } A\mathbf{x} + B\mathbf{y} = \mathbf{c}$$

Two blocks of variables with separable objectives

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Two blocks of variables with separable objectives
The augmented Lagrangian is

$$L_\rho(x, y, u) = f(x) + g(y) + u^T(Ax + By - c) + \frac{\rho}{2}||Ax + By - c||_2^2$$

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Applying the method of multipliers, we need to find (x_k, y_k) by solving the joint minimization:

$$(x_k, y_k) = \arg\min_{x,y} L_\rho(x, y, u_{k-1})$$

Alternating Direction Method of Multipliers (ADMM)

ADMM splits the joint minimization into two steps :

$$x_k = \arg\min_x L_\rho(x, y_{k-1}, u_{k-1}) \tag{7}$$

$$y_k = \arg\min_y L_\rho(x_k, y, u_{k-1}) \tag{8}$$

$$u_k = u_{k-1} + \rho(Ax_k + By_k - c) \tag{9}$$

$\Longrightarrow$ Splitting since we minimize over x with y fixed, and vice versa (**Gauss-Seidel method**)

Example

$$\begin{cases} \min_{\mathbf{x}_i} \sum_{i=1}^{N} f_i(\mathbf{x}_i) \\ \text{s.t } \mathbf{x}_i = \mathbf{x} \quad i = 1 \dots N \end{cases}$$

The augmented Lagrangian

$$L_\rho(x, y, u) = \sum_{i=0}^{N} f_i(x_i) + u^T(x_i - x) + \frac{\rho}{2}||x_i - x||_2^2$$

ADMM iterates

$$x_{k+1}^i = \arg\min_{x_i}(f_i(x_i) + (u_k^i)^T x_i + \frac{\rho}{2}||x_i - x_k||_2^2)$$

$$x_{k+1} = \frac{1}{N}\sum_{i=0}^{N} x_{k+1}^i + \frac{1}{\rho}u_k^i$$

$$u_{k+1}^i = u_k^i + \rho(x_{k+1}^i - x_{k+1})$$

Convergence guarantees

Under modest assumptions on f, g, the ADMM iterates satisfy, for any $\rho > 0$ [4]:

- Residual convergence/Primal feasibility: $r_k = Ax_k + By_k - c \longrightarrow 0$ as $k \longrightarrow \infty$
- Primal convergence: $f(x_k) + g(y_k) \longrightarrow f^* + g^*$
- Dual convergence: $u_k \longrightarrow u^*$

Assumption 1. The (extended-real-valued) functions $f : \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ and and $g : \mathbb{R}^m \mapsto \mathbb{R} \cup \{+\infty\}$ are closed, proper, and convex. $\implies$ epif $= \{(x, t) \in \mathbb{R}^n \times \mathbb{R} | f(x) \leq t\}$ is a closed nonempty convex set.

Assumption 2. The unaugmented Lagrangian L has a saddle point.

Note: No further assumptions on A and B

Convergence rate: Theory still being developed but ADMM has similar convergence rate as 1st order methods (linear) (i.e [5] linear convergence rate when one of the functions is strongly convex)

Scaled form ADMM

Let the residual $r = Ax + Bz - c$ and the scaled dual variable $w = \frac{u}{\rho}$

$$u^T r + \frac{\rho}{2}||r||_2^2 = \frac{\rho}{2}||r + \frac{1}{\rho}u||_2^2 - \frac{2}{2\rho}||u||_2^2 = \frac{\rho}{2}||r + w||_2^2 - \frac{\rho}{2}||w||_2^2$$

The augmented Lagrangian becomes

$$L_\rho(x, y, w) = f(x) + g(y) + \frac{\rho}{2}||r + w|| - \frac{\rho}{2}||w||_2^2$$

And the ADMM iterates

$$x_{k+1} = \arg\min_x f(x) + \frac{\rho}{2}||Ax + By_k - c + w_k||_2^2$$

$$y_{k+1} = \arg\min_y g(y) + \frac{\rho}{2}||Ax_{k+1} + By - c + w_k||_2^2$$

$$w_{k+1} = w_k + Ax_{k+1} + By_{k+1} - c$$

The residual at iteration k, $r_k = Ax_k + By_k - c$,

$$w_{k+1} = w_0 + \sum_{i=1}^{k} r_j \longrightarrow \text{ running sum of residuals}$$

Douglas–Rachford algorithm

Consider the problem, where f and g are closed convex functions

$$\min_x f(x) + g(x)$$

Douglas–Rachford iteration [6]:

$$\begin{aligned} y_{k+1} &= y_k + \mathrm{prox}_g(2x_k - y_k) - x_k \\ x_{k+1} &= \mathrm{prox}_f(y_{k+1}) \end{aligned}$$

An equivalent formulation

$$\begin{aligned} u_{k+1} &= \mathrm{prox}_g(2x_k - y_k) \\ x_{k+1} &= \mathrm{prox}_f(y_k + u_{k+1} - x_k) \\ y_{k+1} &= y_k + u_{k+1} - x_{k+1} \end{aligned}$$

Let $w_k = x_k - y_k$

$$\begin{aligned} u_{k+1} &= \mathrm{prox}_g(x_k + w_k) \\ x_{k+1} &= \mathrm{prox}_f(u_{k+1} - w_k) \\ w_{k+1} &= w_k + x_{k+1} - u_{k+1} \end{aligned}$$

Douglas–Rachford algorithm and ADMM

ADMM is equivalent to applying Douglas–Rachford in the Dual :

$$\min_{x,y} f(x) + g(y) \text{ s.t } Ax + By = c$$

Dual Problem: $\max_z -c^T z - f^\star(-A^T z) - g^\star(-B^T z)$
Apply the Douglas–Rachford method to minimize
$\underbrace{c^T z + f^\star(-A^T z)}_{f_1(z)} + \underbrace{g^\star(-B^T z)}_{f_2(z)}$

$$u_{k+1} = \text{prox}_{f_1}(z_k + w_k) \tag{10}$$

$$z_{k+1} = \text{prox}_{f_2}(u_{k+1} - w_k) \tag{11}$$

$$w_{k+1} = w_k + z_{k+1} - u_{k+1} \tag{12}$$

Recall that the PPA algorithm in the dual is equivalent to the augmented Lagrangian in the primal

Douglas–Rachford algorithm and ADMM

(10) $\Longrightarrow$

$$\tilde{x} = \arg\min_{x} f(x) + (z_k + w_k)^T(Ax - b) + \rho/2||Ax - b||_2^2 \tag{13}$$

$$u_{k+1} = (z_k + w_k) + \rho(A\tilde{x} - b)$$

(11) $\Longrightarrow$

$$\tilde{y} = \arg\min_{y} f(y) + (z_k)^T(By) + \rho/2||A\tilde{x} + By - b||_2^2 \tag{14}$$

$$z_{k+1} = z_k + \rho(A\tilde{x} + B\tilde{y} - b) \tag{15}$$

From (13), (14) and (15), we can derive the ADMM iterates

ADMM in practice

- Modest accuracy in a handful of iterations, but it requires a large number of iterations for a highly accurate solution (like a first-order method)
- Choice of ρ can greatly influence practical convergence of ADMM:
  - ρ too large $\longrightarrow$ not enough emphasis on minimizing $f + g$
  - ρ too small $\longrightarrow$ not enough emphasis on feasibility
  - Vector Approximate Message Passing (VAMP) find the optimal ρ value under the Gaussian approximation of the messages
- Transforming a problem to ADMM format can be subtle and leads to different algorithms

ADMM for constrained convex optimization

Consider problem

$$\min_x f(x) \text{ s.t } x \in C$$

Transform the problem to ADMM format

$$\min_{x,y} f(x) + g(y) \text{ s.t } x - y = 0$$

where $g = \mathbb{I}_C$

$$x_{k+1} = \arg\min_x \left(f(x) + \frac{\rho}{2}||x - y_k + u_k||_2^2\right)$$

$$y_{k+1} = \arg\min_y \left(g(y) + \frac{\rho}{2}||x_{k+1} + u_k - y||_2^2\right) = \Pi_C(x_{k+1} + u_k)$$

$$u_{k+1} = u_k + x_{k+1} - y_{k+1}$$

ADMM for computer vision

Image Deblurring[7]

original noisy/blurred restored

Sparse + Low rank matrix decomposition[8]

(a) Original frames (b) Low-rank $\hat{L}$ (c) Sparse $\hat{S}$

Table of Contents

Consensus ADMM

Consider the problem

$$\min_{x} \sum_{i=0}^{N} f_i(x)$$

The **Consensus ADMM** starts by transforming the problem to

$$\min_{x_i, x} \sum_{i=0}^{N} f_i(x_i) \text{ s.t } x_i = x, \quad i = 1 \ldots N$$

Thus, the decomposoble ADMM iterates

$$x_k^i = \arg\min_{x_i} f_i(x_i) + \rho/2||x_i - x_{k-1} + w_{k-1}^i||_2^2 \quad i = 1, \ldots, N$$

$$x_k = 1/N \sum_{i=0}^{N} (x_k^i + w_{k-1}^i)$$

$$w_k^i = w_{k-1}^i + x_k^i - x_k, \quad i = 1, \ldots, N$$

Consensus ADMM

By taking $x_k = \bar{x} = 1/N \sum_{i=0}^{N} x_k^i$, the previous iterates can be simplified to

$$x_k^i = \arg\min_{x_i} f_i(x_i) + \rho/2||x_i - \bar{x}_{k-1} + w_{k-1}^i||_2^2 \quad i = 1, \ldots, N$$

$$w_k^i = w_{k-1}^i + x_k^i - \bar{x}_k, \quad i = 1, \ldots, N$$

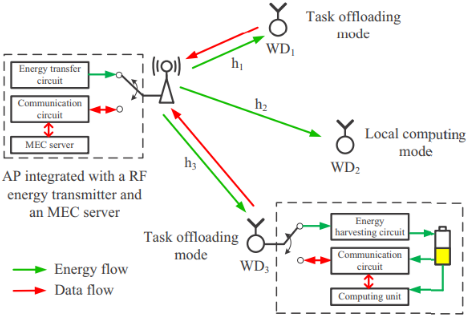
- the $x_i, i = 1, \ldots, B$ updates are done **in parallel**
- Consensus ADMM is **communication inefficient** since we need to gather all the x_k^i to update x_k and thus w_k^i
  Recently, Group ADMM (GADMM) [9] is proposed to tackle this problem, where a worker i only communicates to two neighbors

$$\min_{x_i} \sum_{i=0}^{N} f_i(x_i) \text{ s.t } x_i = x_{i+1}, \quad i = 1 \ldots N$$

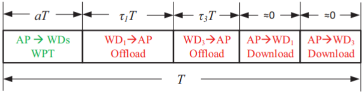
Task Offloading in MEC [10]

- MEC newtork with an AP and N wireless devices (WD)
- AP brodcasts RF energy to distributed WDs and the WDs harvest that energy for aT time, $a \in [0, 1]$ in each time frame
- Within each time frame, we assume that each WD needs to accomplish a certain computing task (pollution level) based on its local data (measurements)
- WD decides to compute the task locally (mode 0) or offload it to the AP (mode 1) (binary offloading)
- $M = M_0 \cup M_1 = \{1 \dots N\}$ the set of WDs
- Harvested energy $E_i = \mu P h_i a T, i = 1 \dots N$, P is RF energy transmit power of the AP, $\mu \in [0, 1]$ is the energy harvesting efficiency, h_i is the channel gain between the AP and the i-th WD
- Transmission time $\tau_i T$, $\tau \in [0, 1]$

Task Offloading in MEC



An example 3-user wireless powered MEC system with binary computation offloading



An example time allocation in the 3-user wireless powered MEC network. Only WD1 and WD3 selecting mode 1 offload the task to and download the computation results from the AP

- **Local computation model**

$$r_i = \nu_1 (\frac{h_i}{k_i})^{1/3} a^{1/3} \tag{16}$$

where r_i is the local computation rate (bits/s), k_i denotes the computation energy efficiency coefficient of the processor's chip and ν_1 is a fixed parameter

- **offloading mode**

$$r_i = \frac{B\tau_i}{\nu_u} \log_2(1 + \frac{\mu P a h_i^2}{\tau_i N_0}) \forall i \in M_1 \tag{17}$$

where ν_u indicates the communication overhead in task offloading, B the bandwidth and N_0 the noise power

Problem formulation

$$
\begin{aligned}
(P1): \underset{\mathcal{M}_0, a, \boldsymbol{\tau}}{\text{maximize}} \quad & \sum_{i \in \mathcal{M}_0} w_i \eta_1 \left(\frac{h_i}{k_i}\right)^{\frac{1}{3}} a^{\frac{1}{3}} && \text{(7a)} \\
& + \sum_{j \in \mathcal{M}_1} w_j \varepsilon \tau_j \ln\left(1 + \frac{\eta_2 h_j^2 a}{\tau_j}\right) && \text{(7b)} \\
\text{subject to} \quad & \sum_{j \in \mathcal{M}_1} \tau_j + a \leq 1, && \text{(7c)} \\
& a \geq 0, \ \tau_j \geq 0, \ \forall j \in \mathcal{M}_1, && \text{(7d)} \\
& \mathcal{M}_0 \subseteq \mathcal{M}, \ \mathcal{M}_1 = \mathcal{M} \setminus \mathcal{M}_0. && \text{(7e)}
\end{aligned}
$$

ADMM formulation

$$
\begin{aligned}
\underset{a,\mathbf{z},\mathbf{x},\boldsymbol{\tau},\mathbf{m}}{\text{maximize}} \quad & \sum_{i=1}^{N} w_i \Bigg\{ (1-m_i)\,\eta_1 \left(\frac{h_i}{k_i}\right)^{\frac{1}{3}} x_i^{\frac{1}{3}} \\
& + m_i \varepsilon \tau_i \ln\left(1+\frac{\eta_2 h_i^2 x_i}{\tau_i}\right) \Bigg\} \\
\text{subject to} \quad & \sum_{i=1}^{N} z_i + a \le 1, \\
& x_i = a,\, z_i = \tau_i \;\; i = 1,\cdots,N, \\
& a, z_i, x_i, \tau_i \ge 0,\;\; m_i \in \{0,1\},\;\; i = 1,\cdots,N.
\end{aligned} \tag{8}
$$

Reformulate (P1) as an equivalent integer programming problem by introducing binary decision variables m_i and additional artificial variables x_i and z_i

ADMM formulation

$$\begin{aligned} \underset{a,\mathbf{z},\mathbf{x},\boldsymbol{\tau},\mathbf{m}}{\text{maximize}} \quad & \sum_{i=1}^{N} q_i(x_i, \tau_i, m_i) + g(\mathbf{z}, a) && \text{(9a)} \\ \text{subject to} \quad & x_i = a, \tau_i = z_i \ i = 1, \cdots, N, && \text{(9b)} \\ & x_i, \tau_i \geq 0, \ m_i \in \{0,1\}, \ i = 1, \cdots, N, && \text{(9c)} \end{aligned}$$

where

$$q_i(x_i, \tau_i, m_i) = w_i \left\{ (1 - m_i)\, \eta_1 \left(\frac{h_i}{k_i} \right)^{\frac{1}{3}} x_i^{\frac{1}{3}} + m_i \varepsilon \tau_i \ln \left(1 + \frac{\eta_2 h_i^2 x_i}{\tau_i} \right) \right\},$$

and

$$g(\mathbf{z}, a) = \begin{cases} 0, & \text{if } (\mathbf{z}, a) \in \mathcal{G}, \\ -\infty, & \text{otherwise,} \end{cases} \qquad (10)$$

where

$$\mathcal{G} = \left\{ (\mathbf{z}, a) \mid \sum_{i=1}^{N} z_i + a \leq 1, a \geq 0, z_i \geq 0, i = 1, \cdots, N \right\}.$$

Notice that $i \in M_0$ can be removed from the constraint set

ADMM formulation

$$L\left(\mathbf{u},\mathbf{v},\boldsymbol{\theta}\right)=\sum_{i=1}^{N} q_i(\mathbf{u})+g(\mathbf{v})+\sum_{i=1}^{N}\beta_i\left(x_i-a\right)$$
$$+\sum_{i=1}^{N}\gamma_i\left(\tau_i-z_i\right)-\frac{c}{2}\sum_{i=1}^{N}\left(x_i-a\right)^2-\frac{c}{2}\sum_{i=1}^{N}\left(\tau_i-z_i\right)^2$$

ADMM formulation

$$L(\mathbf{u}, \mathbf{v}, \boldsymbol{\theta}) = \sum_{i=1}^{N} q_i(\mathbf{u}) + g(\mathbf{v}) + \sum_{i=1}^{N} \beta_i (x_i - a)$$
$$+ \sum_{i=1}^{N} \gamma_i (\tau_i - z_i) - \frac{c}{2} \sum_{i=1}^{N} (x_i - a)^2 - \frac{c}{2} \sum_{i=1}^{N} (\tau_i - z_i)^2$$

Step 1: maximize w.r.t to $u = \{x, \tau, m\}$

$$u_{t+1} \arg\max_{u} L(u, v_t, \theta_t) \tag{18}$$

ADMM formulation

$$L(\mathbf{u}, \mathbf{v}, \boldsymbol{\theta}) = \sum_{i=1}^{N} q_i(\mathbf{u}) + g(\mathbf{v}) + \sum_{i=1}^{N} \beta_i (x_i - a)$$
$$+ \sum_{i=1}^{N} \gamma_i (\tau_i - z_i) - \frac{c}{2} \sum_{i=1}^{N} (x_i - a)^2 - \frac{c}{2} \sum_{i=1}^{N} (\tau_i - z_i)^2$$

Step 1: maximize w.r.t to $u = \{x, \tau, m\}$

$$u_{t+1} \arg\max_{u} L(u, v_t, \theta_t) \tag{18}$$

$$\begin{cases} \underset{x_i, \tau_i \geq 0}{\text{maximize}} & w_i \eta_1 \left(\frac{h_i}{k_i}\right)^{\frac{1}{3}} x_i^{\frac{1}{3}} + \beta_i^l x_i + \gamma_i^l \tau_i - \frac{c}{2}\left(x_i - a^l\right)^2 \\ & -\frac{c}{2}\left(\tau_i - z_i^l\right)^2, \text{ if } m_i = 0, \\ \underset{x_i, \tau_i \geq 0}{\text{maximize}} & w_i \varepsilon \tau_i \ln\left(1 + \frac{\eta_2 h_i^2 x_i}{\tau_i}\right) + \beta_i^l x_i + \gamma_i^l \tau_i \\ & -\frac{c}{2}\left(x_i - a^l\right)^2 - \frac{c}{2}\left(\tau_i - z_i^l\right)^2, \text{ if } m_i = 1. \end{cases}$$

ADMM Algorithm

$$L\left(\mathbf{u},\mathbf{v},\boldsymbol{\theta}\right)=\sum_{i=1}^{N}q_i(\mathbf{u})+g(\mathbf{v})+\sum_{i=1}^{N}\beta_i\left(x_i-a\right)$$

$$+\sum_{i=1}^{N}\gamma_i\left(\tau_i-z_i\right)-\frac{c}{2}\sum_{i=1}^{N}\left(x_i-a\right)^2-\frac{c}{2}\sum_{i=1}^{N}\left(\tau_i-z_i\right)^2$$

ADMM Algorithm

$$L(\mathbf{u}, \mathbf{v}, \boldsymbol{\theta}) = \sum_{i=1}^{N} q_i(\mathbf{u}) + g(\mathbf{v}) + \sum_{i=1}^{N} \beta_i (x_i - a) + \sum_{i=1}^{N} \gamma_i (\tau_i - z_i) - \frac{c}{2} \sum_{i=1}^{N} (x_i - a)^2 - \frac{c}{2} \sum_{i=1}^{N} (\tau_i - z_i)^2$$

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ADMM Algorithm

Step 2: maximize w.r.t to $v = \{z, a\}$

$$u_{t+1} \arg\max_{u} L(u_{t+1}, v, \theta_t) \tag{20}$$

ADMM Algorithm

Step 2: maximize w.r.t to $v = \{z, a\}$

$$u_{t+1} \arg\max_{u} L(u_{t+1}, v, \theta_t) \tag{20}$$

$$\begin{aligned}
&\mathbf{v}^{l+1} = \\
&\underset{\mathbf{z},a}{\arg\text{maximize}} \quad \sum_{i=1}^{N} \beta_i^l \left(x_i^{l+1} - a\right) + \sum_{i=1}^{N} \gamma_i^l \left(\tau_i^{l+1} - z_i\right) \\
&\qquad\qquad - \frac{c}{2} \sum_{i=1}^{N} \left(x_i^{l+1} - a\right)^2 - \frac{c}{2} \sum_{i=1}^{N} \left(\tau_i^{l+1} - z_i\right)^2 \\
&\text{subject to} \quad \sum_{i=1}^{N} z_i + a \leq 1, \;\; a \geq 0, \;\; z_i \geq 0, i = 1, \cdots, N.
\end{aligned}$$

ADMM Algorithm

Step 2: maximize w.r.t to $v = \{z, a\}$

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&\qquad\qquad - \frac{c}{2} \sum_{i=1}^{N} \left(x_i^{l+1} - a\right)^2 - \frac{c}{2} \sum_{i=1}^{N} \left(\tau_i^{l+1} - z_i\right)^2 \\
&\text{subject to} \quad \sum_{i=1}^{N} z_i + a \leq 1, \;\; a \geq 0, \;\; z_i \geq 0, i = 1, \cdots, N.
\end{aligned}$$

Step 3: minimize w.r.t to $\theta = \{\beta, \gamma\}$

$$\begin{aligned}
\beta_i^{l+1} &= \beta_i^l - c(x_i^{l+1} - a^{l+1}), \;\; i = 1, \cdots, N, \\
\gamma_i^{l+1} &= \gamma_i^l - c(\tau_i^{l+1} - z_i^{l+1}), \;\; i = 1, \cdots, N.
\end{aligned}$$

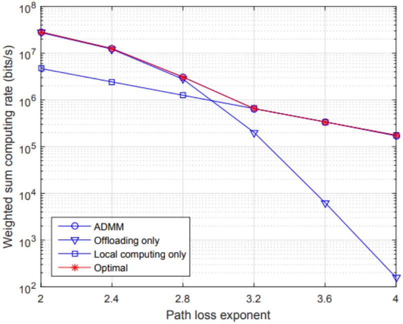
Results I

Baselines

- **Optimal**: exhaustively enumerates all the 2^N combinations of N WDs' computing modes;
- **Offloading only**: all the WDs offload their tasks to the AP, $M_0 = \{\}$;
- **Local computing only**: all the WDs perform computations locally, $M_0 = M$.

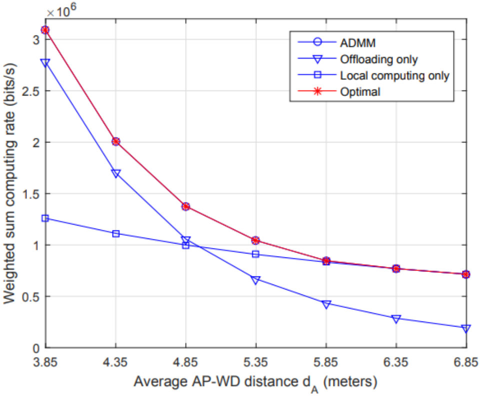
Baselines

- **Optimal**: exhaustively enumerates all the 2^N combinations of N WDs' computing modes;
- **Offloading only**: all the WDs offload their tasks to the AP, $M_0 = \{\}$;
- **Local computing only**: all the WDs perform computations locally, $M_0 = M$.



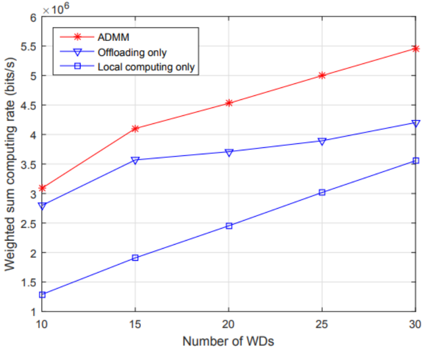
Under different path loss exponents (N=10)

Results II



Under different average AP-WD distance (N=10)

Results III


×10^6
ADMM
Offloading only
Local computing only
Weighted sum computing rate (bits/s)
Number of WDs
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
10
15
20
25
30

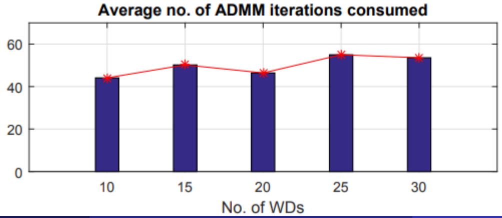

Average no. of ADMM iterations consumed
60
40
20
0
10
15
20
25
30
No. of WDs

Table of Contents

- Decomposition is necessary for decentralized optimization
- Coordination between sub-problems can be costly and communication inefficient
- ADMM offers good convergence guarantees with mild assumptions on the objective functions
- ADMM may result in different algorithms when the problem is formulated differently

What we didn't cover

- More on consensus algorithms
- Recent work on distributed gradient algorithms

Lecture Notes I

- ECE236C - Optimization Methods for Large-Scale Systems, Prof. L. Vandenberghe, UCLA
- EE364b - Convex Optimization II, Prof. S. Boyd, Stanford
- 10-725: Convex Optimization, Prof. R. Tibshirani, CMU

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